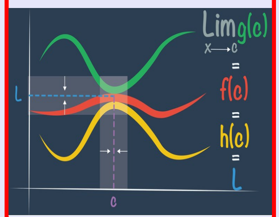
Calculus I

Lecture 7



Feb 19-8:47 AM

Class QZ 6

For $\varepsilon > 0$, find a $\delta > 0$ such that $\lim_{x \to 5} x^2 = 25$. ✓

$f(x) = x^2$

$a = 5$

$L = 25$ ✓

$\lim_{x \to 5} x^2 = 5^2 = 25$ ✓

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|x^2 - 25| < \varepsilon$ whenever $|x - 5| < \delta$

$|(x+5)(x-5)| < \varepsilon$

$\underbrace{|x+5|}_{\text{Bound}} \underbrace{|x-5|}_{\text{Keep}} < \varepsilon$

If $|x+5| < C$

$C|x-5| < \varepsilon$

$|x-5| < \frac{\varepsilon}{C}$

$\boxed{\delta = \min\left\{1, \frac{\varepsilon}{11}\right\}}$

Since $f(x) = x^2$ is a Polynomial function

It is common to have $\delta < 1$

$|x-5| < 1$

$-1 < x - 5 < 1$

Add 10

$-1 + 10 < x - 5 + 10 < 1 + 10$

$9 < x + 5 < 11$

$|x+5| < 11$

Jun 25-7:38 AM

Given $f(x)=\begin{cases} x^2+1 & \text{if } x<1 \\ (x-2)^2 & \text{if } x\geq 1 \end{cases}$

1) $\lim_{x\to 1^-} f(x)=(1)^2+1$ $=\boxed{2}$

$x<1$

2) $\lim_{x\to 1^+} f(x)=(1-2)^2$ $=(-1)^2=\boxed{1}$

$x>1$

3) $\lim_{x\to 1} f(x)$ $\boxed{\text{D.N.E.}}$

4) $f(1)=(1-2)^2$ $=\boxed{1}$

5) Is $f(x)$ continuous at $x=1$? Explain.

NO

$\lim_{x\to 1} f(x)$ D.N.E.

$\therefore \lim_{x\to 1} f(x) \neq f(1)$

Jun 26-8:21 AM

Prove $\lim_{x\to 0} x^4 \cos\frac{2}{x}=0$

Hint:

$-1\leq \cos\alpha \leq 1$

$-1\leq \cos\frac{2}{x} \leq 1$

multiply by x^4

Since $x^4\geq 0$, it does not change the inequalities.

$-x^4 \leq x^4\cos\frac{2}{x} \leq x^4$

$\lim_{x\to 0}(-x^4)=-0^4=0$

$\lim_{x\to 0} x^4=0^4=0$

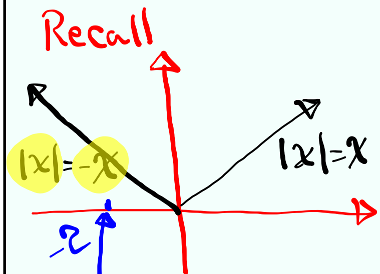
By S.T.

$\lim_{x\to 0} x^4\cos\frac{2}{x}=\boxed{0}$

Jun 26-8:30 AM

Evaluate $\lim\limits_{x \to -2} \dfrac{2-|x|}{2+x} = \lim\limits_{x \to -2} \dfrac{2-(-x)}{2+x}$

Recall

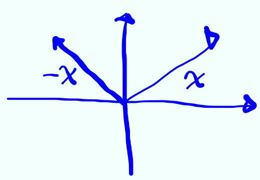


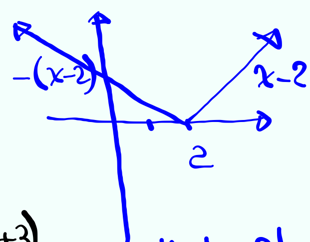
$= \lim\limits_{x \to -2} \dfrac{2+x}{2+x}$

$= \lim\limits_{x \to -2} 1 = \boxed{1}$

Jun 26-8:38 AM

$f(x) = \dfrac{x^2+x-6}{|x-2|}$





1) $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} \dfrac{x^2+x-6}{x-2}$

$x > 2$

$= \lim\limits_{x \to 2^+} \dfrac{(x+3)\cancel{(x-2)}}{\cancel{x-2}} = \lim\limits_{x \to 2^+} (x+3)$

$= 2+3 = \boxed{5}$

2) $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} \dfrac{x^2+x-6}{-(x-2)}$

$x < 2$

$= -\lim\limits_{x \to 2^-} \dfrac{(x+3)\cancel{(x-2)}}{\cancel{x-2}}$

$= -\lim\limits_{x \to 2^-} (x+3) = \boxed{-5}$

3) $\lim\limits_{x \to 2} f(x)$ D.N.E.

Jun 26-8:44 AM

Given $\lim_{x\to 1} \frac{2f(x)-7}{x-1} = 10$

Find $\lim_{x\to 1} f(x)$.

$$\lim_{x\to 1} \frac{2f(x)-7}{x-1} = \frac{\lim_{x\to 1}[2f(x)-7]}{\lim_{x\to 1}(x-1)} = 10$$

Cross-Multiply

$$\lim_{x\to 1}[2f(x)-7] = 10 \lim_{x\to 1}(x-1)$$

$$\lim_{x\to 1} 2f(x) - \lim_{x\to 1} 7 = 10\cdot(\underbrace{\cancel{1-1}}_{0})$$

$$2\lim_{x\to 1} f(x) - 7 = 0 \qquad 2\lim_{x\to 1} f(x) = 7$$

$$\boxed{\lim_{x\to 1} f(x) = \frac{7}{2}}$$

Jun 26-8:55 AM

Evaluate $\lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} = \frac{\sqrt{6-2}-2}{\sqrt{3-2}-1} = \frac{\sqrt{4}-2}{\sqrt{1}-1} = \frac{0}{0}$ I.F.

$$= \lim_{x\to 2} \frac{(\sqrt{6-x}-2)(\sqrt{6-x}+2)(\sqrt{3-x}+1)}{(\sqrt{3-x}-1)(\sqrt{6-x}+2)(\sqrt{3-x}+1)}$$

$$= \lim_{x\to 2} \frac{(6-x-4)(\sqrt{3-x}+1)}{(3-x-1)(\sqrt{6-x}+2)} = \lim_{x\to 2} \frac{\cancel{(2-x)}(\sqrt{3-x}+1)}{\cancel{(2-x)}(\sqrt{6-x}+2)}$$

$$= \lim_{x\to 2} \frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \frac{\sqrt{3-2}+1}{\sqrt{6-2}+2} = \frac{\sqrt{1}+1}{\sqrt{4}+2} = \frac{2}{4}$$

$$= \boxed{\frac{1}{2}}$$

Jun 26-9:05 AM

Suppose $\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$ exists,

1) Find a

Since the limit exists, we plug in -2.

$$\frac{3(-2)^2 + a(-2) + a + 3}{(-2)^2 + (-2) - 2} = \frac{12 - a + 3}{4 - 2 - 2} = \frac{15 - a}{0} \quad \text{I.F.}$$

If $a = 15$,

2) Find the limit Value.

$$\lim_{x \to -2} \frac{3x^2 + 15x + 18}{x^2 + x - 2} = \frac{0}{0}$$

$$= \lim_{x \to -2} \frac{3(x^2 + 5x + 6)}{(x+2)(x-1)} = \lim_{x \to -2} \frac{3(x+3)\cancel{(x+2)}}{\cancel{(x+2)}(x-1)}$$

$$= \lim_{x \to -2} \frac{3(x+3)}{x-1} = \frac{3(-2+3)}{-2-1} = \frac{3 \cdot 1}{-3} = \boxed{-1}$$

Jun 26-9:27 AM

Evaluate $\lim_{x \to 10} \frac{\sin(x-10)}{x^3 - 1000} = \frac{\sin(10-10)}{10^3 - 1000} = \frac{\sin 0}{0} = \frac{0}{0}$

I.F.

We know $\lim_{h \to 0} \frac{\sin h}{h} = 1$

We also know $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$

$$\lim_{x \to 10} \frac{\sin(x-10)}{x^3 - 1000} = \lim_{x \to 10} \frac{\sin(x-10)}{(x-10)(x^2 + 10x + 100)}$$

$$= \underbrace{\boxed{\lim_{x \to 10} \frac{\sin(x-10)}{x-10}}}_{1} \cdot \boxed{\lim_{x \to 10} \frac{1}{x^2 + 10x + 100}}$$

$$= 1 \cdot \frac{1}{10^2 + 10(10) + 100} = \boxed{\frac{1}{300}}$$

Jun 26-9:40 AM

$$f(x)=\begin{cases}\dfrac{2x^2-5x-3}{x-3} & \text{if } x\neq 3\\ K & \text{if } x=3\end{cases}$$

Find K such that $f(x)$ is continuous at 3.

$$\lim_{x\to 3} f(x) = f(3)$$

$$\boxed{7 = K}$$

$$\lim_{x\to 3}\frac{2x^2-5x-3}{x-3} = \frac{0}{0} \text{ I.F.}$$

$$= \lim_{x\to 3}\frac{(2x+1)\cancel{(x-3)}}{\cancel{x-3}} = \lim_{x\to 3}(2x+1) = 2(3)+1$$

$$= 6+1 = \boxed{7}$$

Jun 26-10:05 AM

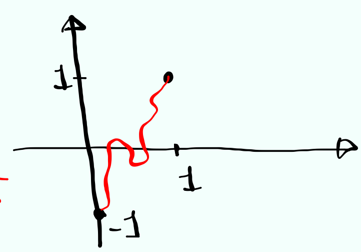
Show $\sqrt[3]{x} = 1-x$ has a solution in $(0,1)$.

$$\boxed{\sqrt[3]{x} - 1 + x} = 0$$

$f(x)=\sqrt[3]{x} - 1 + x \Rightarrow f(x)$ is cont. $(-\infty,\infty)$

$f(0) = \sqrt[3]{0} - 1 + 0 = -1$

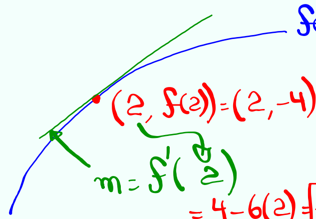
$f(1) = \sqrt[3]{1} - 1 + 1 = 1$



By I.V.T., there is at least one number c in $(0,1)$ such that $f(c)=0$

Jun 26-10:13 AM

Find the equation of the tan. line to the graph of $f(x)=4x-3x^2$ at $x=2$.



$$f(2)=4(2)-3\cdot 2^2 = 8-12 = -4$$

$$m=f'(2) = 4-6(2) = \boxed{-8}$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{4(x+h)-3(x+h)^2-4x+3x^2}{h}=\lim_{h\to 0}\frac{\cancel{4x}+4h\cancel{-3x^2}-6xh-3h^2}{h}$$

$\cancel{-4x+3x^2}$

$$(x+h)^2=(x+h)(x+h)=x^2+2xh+h^2$$

$$=\lim_{h\to 0}\frac{\cancel{h}(4-6x-\cancel{3h}^{\to 0})}{\cancel{h}}$$

$$=\boxed{4-6x}$$

$$y-y_1=m(x-x_1)$$

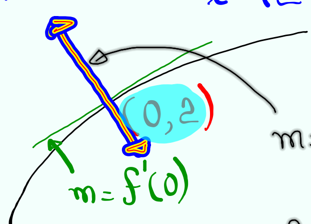
$$y-(-4)=-8(x-2)\rightarrow \boxed{y=-8x+12}$$

Find eqn of the normal line to the graph $f(x)=\dfrac{4}{x+2}$ at $x=0$.

$$y-y_1=m(x-x_1)$$

$$y-2=1(x-0)$$

$$\boxed{y=x+2}$$



$$m=\frac{-1}{f'(0)}=\frac{-1}{\frac{-4}{(0+2)^2}}=\frac{-1}{\frac{-4}{4}}=\frac{-1}{-1}=\boxed{1}$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{4}{x+h+2}-\frac{4}{x+2}}{h}$$

$$=\lim_{h\to 0}\frac{4(x+2)-4(x+h+2)}{h(x+h+2)(x+2)}=\lim_{h\to 0}\frac{-4\cancel{h}}{\cancel{h}(x+h+2)(x+2)}$$

$$LCD=(x+h+2)(x+2)$$

$$=\lim_{h\to 0}\frac{-4}{(x+\cancel{h}^{\to 0}+2)(x+2)}=\boxed{\frac{-4}{(x+2)^2}}$$

Prove $\lim\limits_{h\to 0} \frac{1-\cos h}{h} = 0$ ✓

$\lim\limits_{h\to 0} \frac{\sin h}{h} = 1$

$\lim\limits_{h\to 0} \frac{1-\cos h}{h} = 0$

let's plug it in

$\lim\limits_{h\to 0} \frac{1-\cos h}{h} = \frac{1-\cos 0}{0} = \frac{1-1}{0} = \frac{0}{0}$ I.F.

$A-B \quad A+B$

$\lim\limits_{h\to 0} \frac{1-\cos h}{h} \cdot \frac{1+\cos h}{1+\cos h} = \lim\limits_{h\to 0} \frac{(1-\cos h)(1+\cos h)}{h(1+\cos h)}$

$= \lim\limits_{h\to 0} \frac{1-\cos^2 h}{h(1+\cos h)} = \lim\limits_{h\to 0} \frac{\sin^2 h}{h(1+\cos h)}$

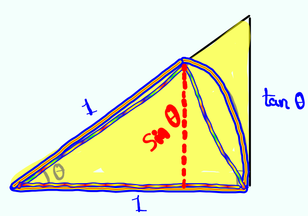
$= \lim\limits_{h\to 0} \left(\frac{\sin h}{h} \cdot \frac{\sin h}{1+\cos h} \right)$

$= \boxed{\lim\limits_{h\to 0} \frac{\sin h}{h}} \cdot \boxed{\lim\limits_{h\to 0} \frac{\sin h}{1+\cos h}}$

$= 1 \cdot \frac{\sin 0}{1+\cos 0} = 1 \cdot \frac{0}{1+1} = 1 \cdot \frac{0}{2}$

$= 1 \cdot 0 = \boxed{0}$

Jun 26-10:38 AM

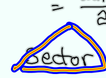


Yellow Area

$A = \frac{1}{2} bh$

$= \frac{1}{2} \cdot 1 \cdot \tan\theta$

$= \frac{\tan\theta}{2}$



Area $= \frac{1}{2} r^2 \theta$

$= \frac{1}{2} \cdot 1^2 \cdot \theta$

$= \frac{\theta}{2}$

Rainbow Area < Sector Area < Yellow Area

$\frac{1}{2} \cdot 1 \cdot \sin\theta < \frac{\theta}{2} < \frac{\tan\theta}{2}$

$\sin\theta < \theta < \tan\theta$

Divide by $\sin\theta$

$1 < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta}$

Do reciprocal

$1 > \frac{\sin\theta}{\theta} > \frac{\sin\theta}{\tan\theta}$

$\frac{\sin\theta}{\tan\theta} < \frac{\sin\theta}{\theta} < 1$

$\cos\theta < \frac{\sin\theta}{\theta} < 1$

take limit as $\theta \to 0$

$\lim\limits_{\theta\to 0} \cos\theta < \lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} < \lim\limits_{\theta\to 0} 1$

$\cos 0 < \lim \frac{\sin\theta}{\theta} < 1$

$1 < \lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} < 1$

By S.T.

$\boxed{\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1}$

Jun 26-10:47 AM